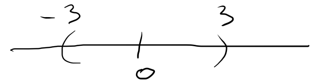
Mocninné řady

Nalezněte středy a poloměry konvergence mocninných řad:

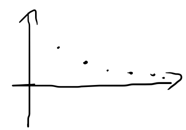
- $\sum_{n=0}^{\infty} \frac{x^n}{2^n+3^n}$ střed je $x_0 = 0$



$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}+3^{n+1}}\cdot\frac{2^n+3^n}{x^n}\right| = \lim_{n\to\infty} |x| \cdot \frac{3^n\left[\left(\frac{2}{3}\right)^n+1\right]}{3^{n+1}\left[\left(\frac{2}{3}\right)^{n+1}+1\right]} = \frac{|x|}{3} < 1$$

řada ABS. konv. pro $|x| < 3$

$\left(\frac{2}{3}\right)^n$

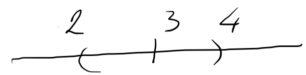


$\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

pro $x = \pm 3$ řada nekonv.

$\lim_{n\to\infty} a_n \neq 0$

- $\sum_{k=0}^{\infty} \frac{(x-3)^k}{(k+5)^2}$ stř. $x_0 = 3$



$$\lim_{n\to\infty} \frac{|x-3|^{k+1}}{(k+1+5)^2} \cdot \frac{(k+5)^2}{|x-3|^k} = |x-3| < 1$$

řada je ABS. konv. pro $x \in \langle 2, 4 \rangle$

pro $x = 2$ $\sum_{k=0} \frac{(-1)^k}{(k+5)^2}$ ABS. KONV.

pro $x = 4$ $\sum_{k=0}^{\infty} \frac{1}{(k+5)^2}$ ABS. KONV.

Najděte Taylorův rozvoj funkce na okolí zadaného bodu:

- $x\cos(5x)$ střed $x_0 = 0$

$$\left[\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right]$$

$$x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{(5x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n} x^{2n+1}}{(2n)!} \qquad R = \infty$$

$$\left[\lim_{n\to\infty} \frac{5^{2(n+1)} |x^{2(n+1)+1}|}{[2(n+1)]!} \cdot \frac{(2n)!}{5^{2n}|x|^{2n+1}} = \lim_{n\to\infty} \frac{25\,|x|^2}{(2n+2)(2n+1)} = 0\right]$$

$$(2n+2)! = (2n+2)(2n+1)(2n)!$$

- $(1+x)\,e^{-x}$ střed $x_0 = -1$

$$\left[e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\right]$$

$$(x+1)\,e^{-[(x+1)-1]} = (x+1)\,e\,e^{-(x+1)} = e(x+1)\cdot\sum_{n=0}^{\infty} \frac{[-(x+1)]^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n e\,(x+1)^{n+1}}{n!}$$

$$R = \infty \quad \left[\lim_{x\to\infty} \frac{|x+1|^{n+2}}{(n+1)!} \cdot \frac{n!}{(x+1)^n} = \lim_{x\to\infty} \frac{|x+1|}{n+1} = 0 \Rightarrow R = \infty\right]$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n, \quad x \in \langle -1, 1);$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R};$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R};$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R};$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \quad x \in (-1, 1\rangle;$$

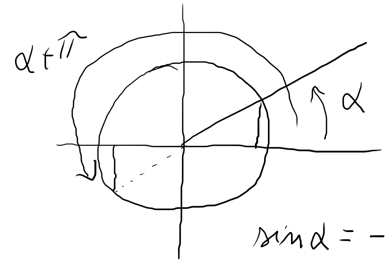
$$(1+x)^\alpha = 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad x \in (-1, 1), \ \alpha \in \mathbb{R}$$

- Pomocí rozvoje exponenciály $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ nalezněte Taylorův rozvoj funkce hyperbolický sinus: $\sinh x = \frac{1}{2}(e^x - e^{-x})$ v bodě $x_0 = -1$ a určete poloměr konvergence.

$$\frac{1}{2}\left(e^{(x+1)-1} - e^{-(x+1)+1}\right) = \frac{1}{2}\left(\frac{1}{e}\sum_{n=0}^{\infty}\frac{(x+1)^n}{n!} - e\sum_{n=0}^{\infty}\frac{(-1)^n(x+1)^n}{n!}\right) =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{e} - e(-1)^n\right)\frac{(x+1)^n}{n!} \qquad R = \infty$$

Najděte Taylorův rozvoj funkce na okolí zadaného bodu:



$\sin\alpha = -\sin(\alpha+\pi)$

- $(\pi - x)\sin x, \qquad x_0 = \pi$

$$-(x-\pi)\sin\left[(x-\pi)+\pi\right] = (x-\pi)\sin(x-\pi) =$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}$$

$$= (x-\pi)\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi)^{2n+1}}{(2n+1)!} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi)^{2n+2}}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ pro } |x|<1$$

- $\dfrac{x+3}{x+2}, \qquad x_0 = 1$

$$\frac{x+3}{x+2} = 1 + \frac{1}{x+2} = 1 + \frac{1}{(x-1)+3} = 1 + \frac{1/3}{1+\frac{x-1}{3}} = 1 + \frac{1}{3}\cdot\frac{1}{1-\left[-\frac{(x-1)}{3}\right]} =$$

pro $\dfrac{|x-1|}{3} < 1$

pro $|x-1|<3$

$$= 1 + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^n}{3^n} = \frac{4}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{(x-1)^n}{3^{n+1}}.$$

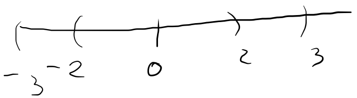
$R = 3$

- $\dfrac{3x-4}{(x+2)(x-3)}$ $\qquad x_0 = 0$

$$\frac{A}{x+2} + \frac{B}{x-3} = \frac{A(x-3)+B(x+2)}{(x+2)(x-3)} \qquad A(x-3)+B(x+2) = 3x-4$$

$$A = 2 \qquad B = 1$$

$$\frac{2}{x+2} + \frac{1}{x-3} = \frac{1}{1+\frac{x}{2}} - \frac{1/3}{1-\frac{x}{3}} = \frac{1}{1-\left(-\frac{x}{2}\right)} - \frac{1/3}{1-\frac{x}{3}} = \left[\begin{array}{l} \text{pro } \left|\frac{x}{2}\right| < 1 \text{ a } \left|\frac{x}{3}\right| < 1 \\ |x| < 2 \text{ a } |x| < 3 \end{array}\right]$$



pro $|x| < 2$

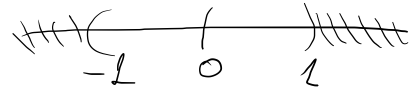
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n} - \frac{1}{3} \sum_{n=0}^{\infty} \frac{x^n}{3^n} =$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2^n} - \frac{1}{3^{n+1}} \right] x^n \qquad R = 2$$

Určete poloměr konvergence a sečtěte mocninnou řadu na vnitřku oboru konvergence.

$$\boxed{\sum_{n=0}^{\infty} n x^{2n+1}}$$

$x_0 = 0$



$$\lim_{n\to\infty} \frac{\left|(n+1)\, x^{2(n+1)+1}\right|}{\left|n\, x^{2n+1}\right|} = \lim_{n\to\infty} \underbrace{\frac{n+1}{n}}_{\to 1} \left| \frac{x^{2n+3}}{x^{2n+1}} \right| = |x^2| < 1, \quad |x| < 1$$

$R = 1$

$$\sum_{n=0}^{\infty} n\, x^{2n+1} = \sum_{n=1}^{\infty} n\, x \cdot (x^2)^n = x \sum_{n=1}^{\infty} n (x^2)^n = x \, \frac{x^2}{(1-x^2)^2} = \boxed{\frac{x^3}{(1-x^2)^2}} \quad x \in (-1,1)$$

$y = x^2$

$$\sum_{n=1}^{\infty} n\, y^n = \sum_{n=1}^{\infty} n\, y \cdot y^{n-1} = y \sum_{n=1}^{\infty} n\, y^{n-1} = y \sum_{n=1}^{\infty} \left[y^n\right]' = y \cdot \left[\sum_{n=1}^{\infty} y^n\right]' =$$

$$= y \cdot \left[\frac{1}{1-y} - 1\right]' = \frac{y}{(1-y)^2}$$

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$$

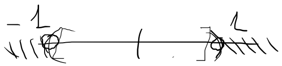
$1 + y + y^2 + \dots$

$|y| < 1$

Určete poloměr konvergence a sečtěte mocninnou řadu na vnitřku oboru konvergence.

$$2n+1-(2n-1)=2$$

$$(-1)^{2n-1} = -1 \qquad \sum_{n=1}^{\infty} \frac{x^{2n-1}}{n} \qquad x_0 = 0$$



$x \in (-1, 1)$

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)-1}}{n+1} \right| \cdot \left| \frac{n}{x^{2n-1}} \right| = \lim_{n\to\infty} \underbrace{\frac{n}{n+1}}_{1} \cdot \left| \frac{x^{2n+1}}{x^{2n-1}} \right| = |x^2| < 1$$

$$|x| < 1 \qquad R = 1$$

pro $x=-1$: $\sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ harm. diverg.

pro $x=1$ $\sum_{n=1}^{\infty} \frac{1}{n}$ harm. diverg.

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}, \quad |y|<1$$

$$y \in (-1,1)$$

$$\sum_{n=1}^{\infty} x^{-1} \frac{(x^2)^n}{n} = \frac{1}{x} \cdot \sum_{n=1}^{\infty} \frac{(x^2)^n}{n} = \frac{-1}{x} \ln(1-x^2) \qquad x \neq 0$$

$$\left[\sum_{n=1}^{\infty} \frac{y^n}{n}\right]' = \sum_{n=1}^{\infty} \left[\frac{y^n}{n}\right]' = \sum_{n=1}^{\infty} y^{n-1} = \sum_{k=0}^{\infty} y^k = \frac{1}{1-y}$$

$$k = n-1$$

$$\sum_{n=1}^{\infty} \frac{y^n}{n} = \int \frac{1}{1-y}\, dy = -\ln(1-y) + C = -\ln(1-y)$$

$|y| < 1$ po dosazení $y=0$ $\quad 0 = -\ln 1 + C \Rightarrow C = 0$

Určete poloměr konvergence a sečtěte mocninnou řadu na vnitřku oboru konvergence.

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$$

$x_0 = 0$

$$\lim_{n\to\infty} \left|\frac{x^{2(n+1)+1}}{2(n+1)+1}\right| \cdot \left|\frac{2n+1}{x^{2n+1}}\right| = \lim_{n\to\infty} \underbrace{\frac{2n+1}{2n+3}}_{\to 1} \cdot \left|\frac{x^{2n+3}}{x^{2n+1}}\right| = |x^2| < 1$$

$|x| < 1 \quad R = 1$

$$\left[\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}\right]' = \sum_{n=0}^{\infty} \frac{\cancel{2n+1}}{\cancel{2n+1}} x^{2n} = \sum_{n=0}^{\infty} (x^2)^n = \frac{1}{1-x^2}$$

$y = x^2$

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y} \qquad |y| < 1$$

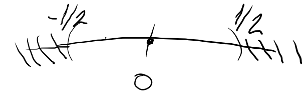
$$= 1 + x^2 + x^4 + \cdots$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = \int \frac{1}{1-x^2}\,dx = \int \left(\frac{1/2}{1-x} + \frac{1/2}{1+x}\right) dx = \frac{1}{2}\left[-\ln(1-x) + \ln(1+x)\right] + \cancel{C}$$

$x = 0 \qquad 0 = \frac{1}{2}(-\ln 1 + \ln 1) + C \quad \Rightarrow \quad C = 0$

$$= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

Určete poloměr konvergence a sečtěte mocninnou řadu na vnitřku oboru konvergence. $R = 1/2$



$\sum_{n=0}^{\infty} b_n$

ABS. KONV.

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = L < 1 \qquad n \in \mathbb{N}$$

$$\sum_{n=0}^{\infty}\left(n+(-2)^n\right)x^n \qquad x_0 = 0$$

$$\lim_{n\to\infty} \frac{|n+1+(-2)^{n+1}|\cdot|x^{n+1}|}{|n+(-2)^n|\cdot|x^n|} = \lim_{n\to\infty} \frac{|(-2)^{n+1}|\cdot\left|\frac{n+1}{(-2)^{n+1}}\to 0 + 1\right|}{|(-2)^n|\cdot\left|\frac{n}{(-2)^n}\to 0 +1\right|} \cdot |x| = |-2|\,|x| = 2|x| < 1$$

$$|x| < 1/2$$

$$R = 1/2$$

$$\sum_{n=0}^{\infty}\left(n+(-2)^n\right)x^n = \sum_{n=0}^{\infty} n\overset{x\cdot x^{n-1}}{x^n} + \sum_{n=0}^{\infty}(-2)^n x^n = \sum_{n=1}^{\infty} n x^n + \sum_{n=0}^{\infty}(-2x)^n$$

$$\boxed{\sum_{n=0}^{\infty} y^n = \frac{1}{1-y} \quad \text{geomet. řada. } |y|<1}$$

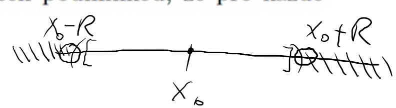
$$= x\sum_{n=0}^{\infty} n x^{n-1} + \sum_{n=0}^{\infty}(-2x)^n \underset{y=-2x}{=}$$

$$= x \cdot \frac{1}{(1-x)^2} + \frac{1}{1-(-2x)} = \frac{x}{(1-x)^2} + \frac{1}{1+2x}, \quad \underline{x \in (-1/2, 1/2)}$$

$$\left[\sum_{n=0}^{\infty} y^n\right]' = \left[1+y+y^2+y^3+\dots\right]' = \left[\frac{1}{1-y}\right]', \quad |y|<1$$

$$\overset{\shortparallel}{\sum_{n=0}^{\infty}}[y^n]' = 1+2y+3y^2+\dots = \sum_{n=0}^{\infty} n y^{n-1} = \frac{1}{(1-y)^2}, \quad |y|<1$$

Poloměr $0 \leq R \leq +\infty$ konvergence mocninné řady $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ je jednoznačně určen podmínkou, že pro každé $x \in \mathbb{R}$ platí



- $|x-x_0| < R \Rightarrow$ řada konverguje (dokonce absolutně),
- $|x-x_0| > R \Rightarrow$ řada diverguje.